Elementary Number Theory Final Exam

Date: 01 November 2022

Place: Exam Hall 1, Aletta Jacobs Hall

Time: 08:30-10:30

Instructions

- To get full points, you must provide complete arguments and computations. You will get no points if you do not explain your answer. Answers like "34", "Yes", "No" will not be accepted.
- While solving a problem, you can use any statement that needs to be proven as a part of another problem even if you did not manage to prove it; e.g. you can use part (a) while solving part (b) even if you did not prove (a). **(In questions 2 and 3, the sub-problems are independent.)**
- Clearly write your name and student number on each page you submit.
- The examination consists of 7 questions. You can score up to 36 points and you get 4 points for free. This way you will score in total between 4 and 40 points.

Problems

1 **(4 points)** Find all integers x such that $x^5 + 3x \equiv -1 \pmod{135}$. (Note that $135 = 5 \cdot 3^3$.)

2 (a) **(3 points)** Prove that $437 \mid 18! + 1$. (Note that $437 = 19 \cdot 23$.)

(b) **(3 points)** Prove that $385 \mid n^{60} - 1$ if n is not divisible by $5, 7, 11$. (Note that $385 = 5 \cdot 7 \cdot 11$.)

3 Recall that $\phi(n)$ denotes the Euler's phi function; $\sigma(n)$ denotes the sum of divisors of n; $\tau(n)$ denotes the number of divisors of n; $\mu(n)$ denotes the Möbius function.

(a) **(3 points)** Show that $\phi(n) > \sqrt{n}/2$ for all positive integers. Use this to conclude that the equation $\phi(n) = k$ has only a finite number of solutions for a fixed k.

(b) **(3 points)** Define $F(n) = \sum_{d \mid n} \mu(d)\sigma(d)$. Compute $F(12!)$.

(c) **(3 points)** Prove that $\tau(n)$ is an odd integer if and only if n is a perfect square.

4 **(3 points)** Show that 3 is a primitive root for every prime $p = 2^n + 1$, where $n > 1$.

5 **(3 points)** Let p and q be odd primes with $p = 4q + a$ for some $a \in \mathbb{Z}$. If $p \equiv 1 \pmod 4$ then show that

$$\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right).$$

6 **(3 points)** Let n be an integer. Show that if $p > 3$ is a prime divisor of $n^2 + 3$ then $p \equiv 1 \mod 3$.

7 (a) **(3 points)** Let $d > 0$ be a non-square integer. Show that if the simple continued fraction expansion of $\sqrt{d}$ has period length 1 then $d = a^2 + 1$ where a is a non-negative integer.

(b) **(3 points)** Let d be a positive integer. Show that the simple continued fraction expansion of $\sqrt{d^2+1}$ is $[d; \overline{2d}]$.

(c) **(2 points)** Describe all solutions $(x, y) \in \mathbb{Z}^2$ of the equation $x^2 - 10y^2 = 1$.

GOOD LUCK! ☺